A LOGARITHMIC COMPLEXITY DIVIDE-AND-CONQUER ALGORITHM FOR MULTI-FLEXIBLE ARTICULATED BODY DYNAMICS

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Abstract. *This paper presents an efficient algorithm for the dynamics simulation and analysis of multi-flexible-body systems. This algorithm formulates and solves the nonlinear equations of motion for mechanical systems with interconnected flexible bodies subject to the limitations of modal superposition, and body substructuring, with arbitrarily large rotations and translations. The large rotations or translations are modelled as rigid body degrees of freedom associated with the interconnecting kinematic joint degrees of freedom. The elastic deformation of the component bodies is modelled through the use of modal coordinates and associated admissible shape functions. Apart from the approximation associated with the elastic deformations, this algorithm is exact, non-iterative and applicable to generalized multi-flexible chain and tree topologies. In its basic form, the algorithm is both time and processor optimal in its treatment of the* n_b *joint variables, providing* $O(log(n_b))$ *turn around time per temporal integration step, achieved with* $O(n_b)$ *processors. The actual cost associated with the parallel treatment of the* n_f *flexible degrees of freedom depends on the specific parallel method chosen for dealing with the individual coefficient matrices which are associated locally with each flexible body.*

${}^{N}\vec{a}^{P}$	Translational acceleration of point P in Newtonian reference frame N
A^{P}	Spatial acceleration, a 6 $\times$ 1 column matrix, of differential element P in Newtonian frame N
A_t^{P}	Spatial acceleration remainder term, a 6 $\times$ 1 column matrix, of differential element P in Newtonian frame N. Contains prescribed motion as well as centripetal and Coriolis accelerations contributions to spatial acceleration.
$A_t^{J^k}$	Spatial acceleration remainder term associated with the acceleration of joint J^k handle k^+ relative to handle k^-
$A_t^{k+1^-/k^+}$	Spatial acceleration remainder term associated with the acceleration of outward handle $k+1^-$ relative to handle k^+, both belonging to body k
$\vec{b}\times$	3$\times$3 skew symmetric matrix for cross product of any vector $\vec{b}$
C^T	Transpose of any arbitrary matrix C
$\mathcal{D}$	Domain of spatial integration
D^{J^k}	Orthogonal complement of P^{J^k}
$\vec{f}^P$	Body force at point *P*
$\vec{f}_c^{\,i}$	Constraint force at joint *i*
$\tilde{f}_c^{\,i}$	Measure numbers of constraint force at joint *i*
$F_c^{\,i}$	Spatial constraint force at joint *i*
J^k	Joint connecting body k to its parent body
$\hat{k}_i$	Unit vector in direction *i*
m^k	Number of degrees-of-freedom associated relative motion across joint J^k
N	Newtonian reference frame
n_b	Number of bodies in the system
P^{J^k}	Free modes of motion subspace map/matrix associated with joint J^k
$P_r^{J^k}$	r^{th} spatial partial velocity of joint handle k^+ relative to handle k^- occurring across joint J^k. The r^{th} column of P^{J^k}
P^{k^+}	Free modes of motion subspace map/matrix associated with absolute motion of handle k^+
$P_r^{k^+}$	r^{th} spatial partial velocity of handle k^+ in Newtonian frame N
q_i^k	i^{th} Modal Coordinate of a flexible body k
$\dot{q}_i^k$	Time derivative of i^{th} modal coordinate of a flexible body k
$\ddot{q}_i^k$	Second time derivative of i^{th} modal coordinate of a flexible body k
$\vec{r}$	Position vector
$S^{k/k+1}$	Shift Matrix between joint k and $k+1$ $\begin{bmatrix} \underline{U} & \vec{r}\times \\ \underline{0} & \underline{U} \end{bmatrix}$
u^{J^k}	Generalized speed matrix associated with motions permitted by joint k
$u_r^{J^k}$	r^{th} Generalized speed associated with motions permitted by joint k. The r^{th} element of u^{J^k}
u^{k^+}	Generalized speed matrix associated with absolute motions of joint handle k^+
$u_r^{k^+}$	r^{th} Generalized speed associated with the absolute motion of handle k^+. The r^{th} element of u^{k^*}

$\dot{u}$	Time derivative of generalized speed u
$\underline{U}$	3×3 Identity matrix
${}^N\vec{v}^P$	Translational velocity of a point P in the Newtonian reference frame N
${}^N\vec{v}_r^{k+}$	r^{th} partial velocity of joint handle k^+ in Newtonian frame N
$\vec{v}_r^{J^k}$	r^{th} partial velocity of joint handle k^+ relative to handle k^- in parent frame
V^{k+}	Absolute spatial velocity, a 6×1 column matrix, of handle (point) k^+ in Newtonian reference frame N
$V_t^{J^k}$	Relative spatial velocity remainder terms, a 6×1 column matrix, associated specified motions which are imposed across joint J^k
$\vec{\alpha}$	Angular acceleration of reference frame in Newtonian frame N
β_{ij}	Contribution to generalized active force
ζ	Inertia coupling terms for individual body or subassembly
ν^k	Number of assumed mode shapes for a flexible body k
ρ	Mass density
$\vec{\tau}_c^i$	Constraint torque at joint i
$\tilde{\tau}_c^i$	Measure numbers of constraint torque at joint i
$\Upsilon^{k:l}$	Inertia coupling terms for assembly comprised of contiguous bodies k through l
$\vec{\varphi}$	Admissible shape function for translational components of deformation
$\vec{\varphi}\|_P$	$\vec{\varphi}$ evaluated at point P
Φ	Spatial matrix containing the shape functions $\begin{bmatrix}\vec{\psi}\\ \vec{\varphi}\end{bmatrix}$
$\vec{\psi}$	Admissible shape function for rotational components of deformation
${}^N\vec{\omega}^P$	Angular velocity of reference frame fixed in differential element P relative to the Newtonian reference frame N
${}^N\vec{\omega}_r^{k+}$	r^{th} partial angular velocity of joint handle k^+ in frame Newtonian frame N
${}^N\vec{\omega}_r^{J^k}$	r^{th} partial angular velocity of joint handle k^+ relative to handle k^-
$\underline{0}$	3×3 Zero matrix

Table 1: The Nomenclature

1 Introduction

Modelling and simulation of the dynamic behavior of complex systems are regularly pursued by engineers and scientists in a wide variety of fields. Applications may include coarse-grained molecular dynamics simulations for advanced material modelling (e.g. polymer chains) and biomolecular systems (e.g. RNA, DNA and proteins); elastic deformation of MEMS devices; modelling the dynamic behavior of multi-continuous bodies (e.g. drive belts, tracks and tracked vehicles); robotic systems and myriad mechanisms and electro-mechanical devices. These systems are typically modelled as articulated, i.e. a system comprising of rigid and (or) flexible bodies interconnected by kinematic joints to form a chain, tree or kinematically closed loop topology. Depending on the system considered and the resolution of the model, these simulations may include a large number of spatial degrees of freedom. For example, applications such as biomolecular systems or molecular modelling of materials may easily involve several thousand ($> 10^5$) spatial degrees of freedom and aim to capture phenomenon over large temporal scales ($> 10^6$ temporal integration steps). With the continued trend towards ever increasing problem size (greater model fidelity, and larger scale systems) and hence growing computational burden, use of efficient algorithms and exploring parallel computing resources have emerged as the primary means to reduce simulation turn-around times.

Due to this accelerating need for understanding, predicting, and controlling the dynamic behavior of many modern engineering systems, the development of algorithms for modelling the dynamic behavior of multibody systems has received considerable attention over the past three decades. These efforts have spawned algorithms and implementation procedures based on fundamentally different philosophies, each with their own strengths and weaknesses. Some of the earliest algorithms were $O(n^3)$ complexity [1]-[2], indicating that the computational cost, which manifests itself in computer CPU time, increases as a cubic function of the number of system generalized coordinates n. Subsequently several algorithms [3]-[9] were independently proposed by various authors for solving the articulated rigid body dynamics problem in $O(n)$ complexity. A brief review of some of these and a discussion of the underlying similarities between different algorithms has been discussed in [9]. These algorithms were initially limited to multi-rigid body applications, but were then extended and generalized to accommodate flexible body systems [10]-[15]. With these so-called $O(n)$ algorithms, the simulation turn around times scale approximately linearly with the increase in system size and hence are more efficient than the traditional $O(n^3)$ approach when applied to large (n>>1) systems.

The lowest computational order possible for articulated body systems when using serial processing is $O(n)$. Parallel processing offers some potential for improving on this. With parallel processing, it becomes theoretically possible to generate and solve the equations of motion of the system in as low as $O(log(n))$ complexity. Actual performance is generally not able to achieve the theoretical complexity due to restrictions of true parallel performance as described by Amdahl's law [16], and actual inter-processor communications costs.

In 1995 Fijany and Sharf [17] proposed the *Constraint Force Algorithm (CFA)* for serial kinematic chains. This was the first parallel algorithm which was time optimal, i.e. $O(log(n))$ in time per temporal integration step and processor optimal, i.e. $O(log(n))$ performance achieved with only $O(n)$ processors. In 1999 Featherstone [18]-[19] proposed a Divide and Conquer Algorithm for articulated rigid body systems (hereafter referred to as *RDCA*). This algorithm also achieved time optimal $O(log(n))$ complexity for parallel implementation with processor optimal $O(n)$ processors, and could be applied to more general topologies. This algorithm is highly efficient for simulating the forward dynamics of articulated rigid body systems when implemented in parallel on number of processors equal to or greater than the number of bodies in the system.

Rigid body dynamics, though able to represent the gross behavior of many systems, is often inadequate in capturing the essential behavior of systems with flexible bodies. In this paper, we present a Divide and Conquer Algorithm for flexible bodies. This algorithm is a generalization of the RDCA to include flexible bodies in the articulated system and maintains logarithmic computational complexity. The goal of this algorithm is to produce and subsequently solve the equations of motion for multi-flexible body systems such that the equations associated with the individual bodies are effectively uncoupled from those of the other system bodies, and are also highly parallelizable.

The individual bodies that form the articulated system are modelled as flexible subject to the limitations of modal superposition and body substructuring, with arbitrarily large rotations and translations. A component mode type formulation is used whereby a reduced set of assumed modes is used to describe the deformation of a component body. These assumed mode shapes could be free-free, clamped-free, or any other appropriate set of vibration modes, constraint modes and / or any required static correction modes. The mode shapes can be obtained for each body from finite element analysis, analytic models or experimental analysis. The choice of the reduced set of shapes has been studied in depth in literature [20]-[22] and without loss of generality it is assumed that a set of admissible mode shapes is obtained for each component body as an input to this algorithm. The temporal coefficients of these mode shapes are treated as the flexible degrees of freedom and solved for in this algorithm. The coupling between the finite joint rotations and the elastic deformations of the bodies is preserved in this formulation. Consequently, the mass matrix obtained for each flexible body is a nonlinear function of the joint and flexible body generalized coordinates associated locally with it, while the bias force terms are nonlinear functions of the generalized coordinates and velocities [24].

2 Theoretical Development of the Algorithm

The theoretical development which follows is divided into two sections. The first deals with the basic Divide and Conquer Algorithm (DCA) which involves the interactions of assemblies with other assemblies through their connecting handles (joints). These assemblies may be point masses, rigid bodies, flexible bodies, or collections of any of these. As such this first portion of the development is independent of the type of components which comprise the assemblies. The subsequent section of the development is dedicated to the efficient handling of aspects which are specific to the treatment of flexible bodies within these assemblies.

2.1 Basic Divide and Conquer Algorithm

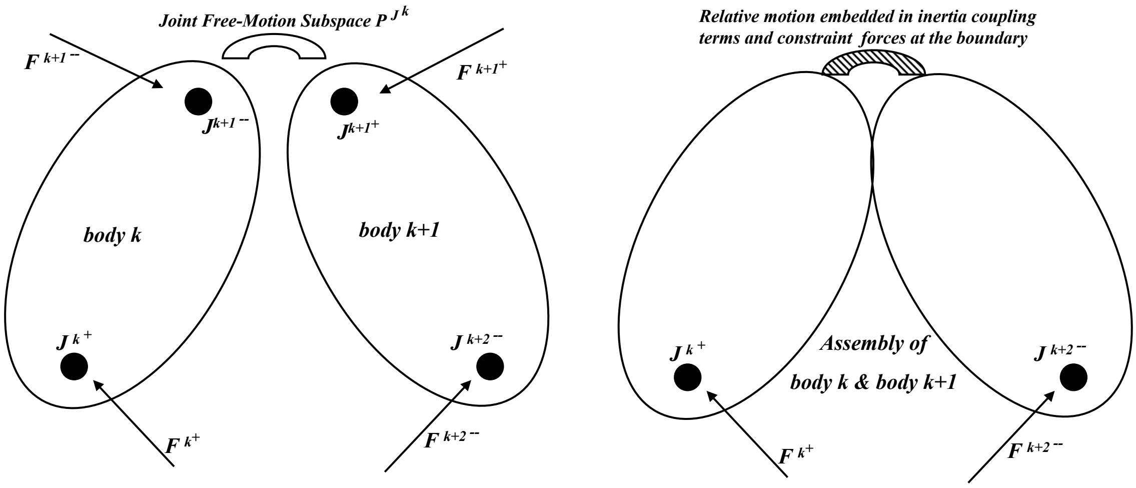
The basic algorithm works in a manner highly similar to the DCA scheme explained in detail in [18][19]. It is presented here so that this paper might be more self contained. The basic unit of the DCA scheme is the two-handle representation of a body. A handle is any selected reference on the body through which the interactions of the body with the environment can be modelled. The handles on a body can correspond to a joint location, a center of mass or any desired reference point. The two handles can even coincide. For the algorithm presented here, the joint locations are chosen as the handles on the body. Consider two representative bodies *Body k* and *Body k+1* of the articulated body system as shown in figure (1A). The two handles on *Body k* correspond to the locations J^{k+} and J^{k+1^-} associated with joints J^k and J^{k+1}, respectively. Similarly, the two handles on *Body k+1* correspond to the locations J^{k+1^+} and J^{k+2^-}, associated with joints J^{k+1} and J^{k+2}. In an effort to make the notation used in the following derivation more concise, points k^+ and k^- will be synonymous with joint J^k handles J^{k+} and J^{k-}, respectively.

There are two main processes in the DCA approach, a hierarchic assembly process and a hierarchic disassembly process. In the hierarchic assembly process, the equations of motion of each body are written in terms of the 6×1 spatial accelerations occurring at each of its two handles (the procedure may be easily generalized to bodies with more than two handles) in the form expressed below for a representative body *Body k*.

$$A^{k^+} = \begin{bmatrix} \vec{\alpha}^{k^+} \\ \vec{a}^{k^+} \end{bmatrix}_{(6\times1)} = \zeta_{11}^k F_c^{k+} + \zeta_{12}^k F_c^{k+1^-} + \zeta_{13}^k \tag{1}$$

$$A^{k+1^-} = \begin{bmatrix} \vec{\alpha}^{k+1^+} \\ \vec{a}^{k+1^+} \end{bmatrix}_{(6\times1)} = \zeta_{21}^k F_c^{k+} + \zeta_{22}^k F_c^{k+1^-} + \zeta_{23}^k \tag{2}$$

The above equation set is henceforth referred to as the two handle equations of motion of representative body *Body k*. Here $\vec{\alpha}^{k^+}$ and $\vec{a}^{k^+}$ represent the 3×1 absolute angular acceleration, and



1A : Representative bodies of an articulated system *1B : Assembly of bodies k and k+1*

Figure 1: Two Handle Articulated Body

3×1 absolute translational acceleration, respectively, of a reference frame fixed in a differential element of volume at point k^+ within the potentially deformable body k. A^{k^+} and A^{k+1^-} are then the spatial accelerations [1] of *Body k* at joint handle locations k^+ and $k+1^-$, respectively. The modal coordinates, as well as their first and second time derivatives manifest themselves in the kinematic relationship which relate accelerations of each of the body's handles. This is shown in detail in section (3.2) which presents the derivation of the equations of motion. The terms $\zeta_{ij}^k\ (i,j = 1,2)$ correspond to inertia coupling at the joint locations on *Body k*. F_c^{k+} and $F_c^{k+1^-}$ are the unknown spatial constraint loads acting on the body at the joints, and are defined as

$$F_c^{k+} = \begin{bmatrix} \vec{\tau}_c^{\,k} \\ \vec{f}_c^{\,k} \end{bmatrix}_{(6\times1)} \qquad \text{and} \qquad F_c^{k+1^+} = \begin{bmatrix} \vec{\tau}_c^{\,k+1} \\ \vec{f}_c^{\,k+1} \end{bmatrix}_{(6\times1)} \tag{3}$$

with $\vec{\tau}_c^{\,i}{}_{(3\times1)}$ and $\vec{f}_c^{\,i}{}_{(3\times1)}$, representing the constraint torques and constraint forces, respectively being imposed through joint i . The active forces at the joint are state dependent and are treated as known quantities. These are lumped together with the state dependent inertia forces and expressed as $\zeta_{ij}\ (i = 1,2;\ \ j = 3)$. Similarly the two handle equations of motion for *Body k+1* can be written in the form

$$A^{k+1^+} = \zeta_{11}^{k+1} F_c^{k+1^+} + \zeta_{12}^{k+1} F_c^{k+2^-} + \zeta_{13}^{k+1} \tag{4}$$

$$A^{k+2^-} = \zeta_{21}^{k+1} F_c^{k+1^+} + \zeta_{22}^{k+1} F_c^{k+2^-} + \zeta_{23}^{k+1} \tag{5}$$

The spatial accelerations A^{k^+} and A^{k^-} occurring at each end of joint J^k are related kinematically by

$$A^{k^+} = A^{k^-} + P^{J^k} \dot{u}^{J^k} + A_t^{J^k} \tag{6}$$

where P^{J^k} is the $6 \times m^k$ matrix of the *free-modes of motion* [23] permitted by the m^k degree-of-freedom joint J^k, and u^{J^k} is the $m^k \times 1$ matrix of associated generalized speeds [28]. These

[1]Note that this definition for spatial Accelerations differs from that presented by Featherstone [18]

free-modes of motion define the manner (both rotational and translational) in which the child body k may move relative to its parent body (in this case body $k-1$). As such P^{J^k} forms a *joint free-motion map* with each column of this matrix being synonymous with the spatial *partial velocities* discussed in [28]. As such the constraint torques and forces which exist through the enforcement of joint constraints, as well as any specified/prescribed joint motions, are orthogonal to these free-modes of motion. Finally, the quantity $A_t^{J^k}$ is the 6×1 matrix of spatial acceleration remainder terms associated with the motion of body k relative to it parent body $k-1$ as permitted by joint J^k. This quantity accounts for all portions of the acceleration of k relative to it parent arising from specified joint motions, as well as Coriolis and centripetal accelerations.

In the hierarchic assembly process, the two handle equations of motion of two successive bodies are coupled together to form the two handle equations of motion of the resulting assembly. If for example we consider the assembly formed from successive bodies k and $k+1$, as shown in figure (1A-B) then the associated assemble two-handle equations are

$$A^{k^+} = \Upsilon_{11}^{k:k+1} F_c^{k+} + \Upsilon_{12}^{k:k+1} F_c^{k+2^-} + \Upsilon_{13}^{k:k+1} \tag{7}$$

$$A^{k+2^-} = \Upsilon_{21}^{k:k+1} F_c^{k+} + \Upsilon_{22}^{k:k+1} F_c^{k+2^-} + \Upsilon_{23}^{k:k+1} \tag{8}$$

The two handles of the resulting assembly are the inward most joint of the *Body k* viz. J^{k+} and the outward most joint on the *Body k+1* viz. J^{k+2^-} and the constraint loads are those acting on the resulting assembly at those handles as indicated in figure (1B). The inertia coupling terms, $\Upsilon_{ij}^{k:k+1}$, for the resulting assembly are calculated using a recursive set of formulae as discussed and presented in section (3.3) of this paper.

This process begins at the level of individual bodies of the system. Adjacent bodies of the system are hierarchically assembled as the construction of a binary tree as shown in figure (2). Individual bodies that make up the system form the leaf nodes of the binary tree. The equations of motion of a pair of bodies are coupled together using the recursive set of formulae to form the two handle equations of motion of the resulting assembly. The resulting assembly now corresponds to a node of the next lower level in the binary tree. The process is then recursively applied further combining adjacent pairs of assemblies into larger (including more component bodies) assemblies as the procedure works downward toward the root of the binary tree. This hierarchic assembly process continues until only a single all-encompassing system assembly is left as the root node of the binary tree. This root node corresponds to the two-handle representation of the entire articulated n-body system reduced to a single assembly, with inertia coupling terms, $\Upsilon_{ij}^{1:n}$, $(i = 1, 2; j = 1, 2, 3)$. The two-handle equations associate with this root node are

$$A^{1^+} = \Upsilon_{11}^{1:n} F_c^{1+} + \Upsilon_{12}^{1:n} F_c^{n+1^-} + \Upsilon_{13}^{1:n} \tag{9}$$

$$A^{n+1^-} = \Upsilon_{21}^{1:n} F_c^{1+} + \Upsilon_{22}^{1:n} F_c^{n+1^-} + \Upsilon_{23}^{1:n} \tag{10}$$

with the two handles of this assembly corresponding to the boundary joints of the articulated system.

If the system is free floating, the spatial constraint forces acting on the two handles are identically zero. The spatial accelerations can then be easily obtained as

$$A^{1^+} = \Upsilon_{13}^{1:n} \quad \text{and} \quad A^{n+1^-} = \Upsilon_{23}^{1:n} \tag{11}$$

If the system is anchored at joint J^1, then the spatial constraint force F_c^{1+} acting on the inward anchored joint is non-zero while at the free end, $F_c^{n+1^-}$ is zero. However, the spatial constraint

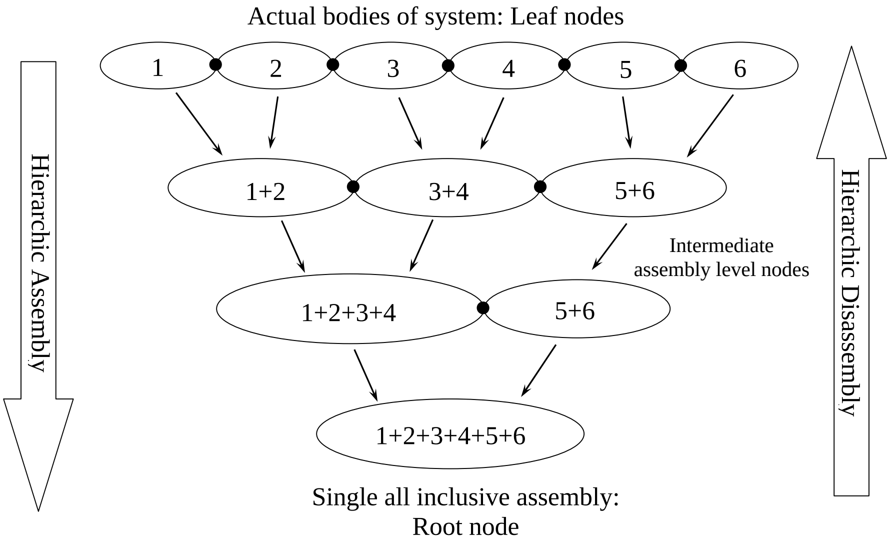


Figure 2: The Hierarchic Assembly and Disassembly Process Using Binary Tree Structure

force F_c^{1+} lies in the subspace orthogonal to the free-modes of motion subspace of joint J^1. Hence the inner product of the matrix P^{J^1} which maps the joint free-modes of motion subspace and the spatial constraint force F_c^{1+} is identically zero. The joint free-modes of motion subspace P^{J^k} and its orthogonal complement D^{J^k} are discussed in further detail in section (3.1). Thus, using the joint free-modes of motion matrix, the two handle equations of motion of the root node can be manipulated as shown below.

$$A^{1^+} = P^{J^1}\dot{u}^{J^1} + A_t^{J^1} = \Upsilon_{11}F_c^{1+} + \Upsilon_{13}^{1:n} \tag{12}$$

$$\Rightarrow (P^{J^1})^T(\Upsilon_{11}^{1:n})^{-1}(P^{J^1}\dot{u}^{J^1} + A_t^{J^1} - \Upsilon_{13}^{1:n}) = (P^{J^1})^T F_c^{1+} = 0 \tag{13}$$

$$\Rightarrow \quad (P^{J^1})^T(\Upsilon_{11}^{1:n})^{-1}P^{J^1}\dot{u}^{J^1} = (P^{J^1})^T(\Upsilon_{11}^{1:n})^{-1}(\Upsilon_{13}^{1:n} - A_t^{J^1}) \tag{14}$$

$$\text{Let} \quad Q = [(P^{J^1})^T(\Upsilon_{11}^{1:n})^{-1}P^{J^1}] \tag{15}$$

$$\Rightarrow \dot{u}^{J^1} = Q^{-1}P^{J^1}(\Upsilon_{11}^{1:n})^{-1}(\Upsilon_{13}^{1:n} - A_t^{J^1}) \tag{16}$$

In the above equations u^{J^1} is the matrix of generalized speeds [28] which characterize the free motions which may take place across joint J^1. Additionally $\Upsilon_{11}^{1:n}$ and Q are symmetric positive definite matrices (SPD) and there is no problem with their matrix inversions. The term $A_t^{J^1}$ is state dependent and is easily calculated. Having solved for $\dot{u}^{J^1}$, the spatial acceleration A^{1^+} can be obtained by equation (6). This value of the spatial acceleration can then be used in two-handle equation (7) to determine the spacial constraint force F_c^{1+}. This inward handle spatial constraint force may then be substituted into the two handle equation (8) for the determination of the spatial acceleration at the outer handle. Thus, whether the system is free floating or anchored, the equations of motion at the root node can be solved to generate the values of the spatial accelerations and spatial constraint forces at the two handles. The case where the system contains kinematically closed loops is the topic of an associated paper [29].

The hierarchic disassembly process begins with the solution of the two-handle equations of motion of the root node. From this solution, the spatial accelerations of and spatial constraint forces acting on the two handles of the single assembly are known. The spatial acceleration and spatial constraint forces generated by solving the two handle equations of an assembly are

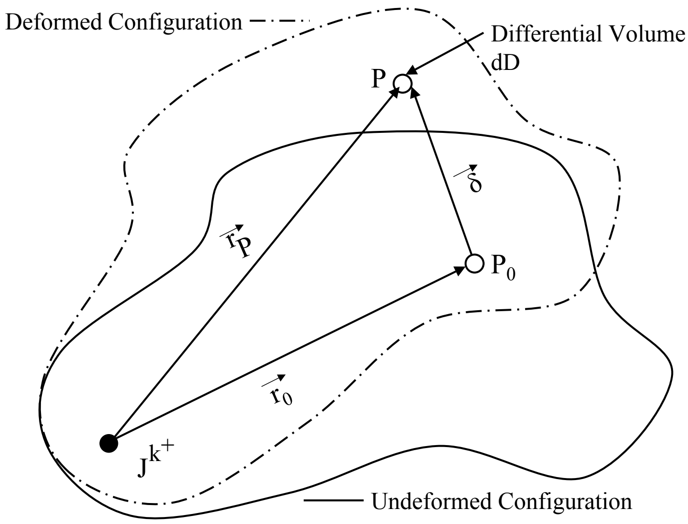


Figure 3: Deformed and Undeformed Configuration of Representative Body k

identically the values of the spatial accelerations and spatial constraint forces on one handle on each of the two constituent assemblies. From these known quantities, the two handle equations of motion of the constituent assemblies can be easily solved for the spatial constraint force and spatial acceleration at the connecting joint. For example, for a representative assembly made from *Body k* and *Body k+1*, the equations of motion are given by equation (7-8). On solving these equations the quantities A^{k^+}, A^{k+2^-}, $F_c^{k^+}$, and $F_c^{k+2^-}$ are generated. These quantities are then substituted into the two-handle equations of the constituent sub-assemblies say for *Body k* and *Body k+1*. Thus knowing the values of A^{k^+}, $F_c^{k^+}$, equations (1-2) can be easily solved, while from A^{k+2^-} and $F_c^{k+2^-}$ equations (4-5) can be solved. This process is repeated in a hierarchic disassembly of the binary tree where the known boundary conditions are used to solve the two-handle equations of motion of the immediate subassemblies, until spatial acceleration and spatial constraint forces on all bodies in the system are calculated. As mentioned above, the time derivatives of the modal coordinates are embedded in the kinematics which relate spatial accelerations of the joints on the same body. Having solved for the spatial accelerations of the joints in the hierarchic disassembly, the time derivatives of the modal coordinates can now be solved for independently on each body. The expression for the time derivatives of the modal coordinates is derived in detail in the section (3.2) which deals with the derivation of the equations of motion.

Similar to the scheme in RDCA, this algorithm works in four sweeps, traversing the system topology like a binary tree. The first and the third sweep work from the leaf nodes of the binary tree to the root node while the second and the fourth sweep work upward, from the root node to the leaf nodes. The input to this algorithm is comprised of the mass properties of the bodies, joint generalized coordinates and speeds, the modal coordinates and their first time derivatives as well as the admissible shape functions for each body. The first two sweeps generate the position and velocity of each handle on each node by using an assembly-disassembly process similar to that described in [18]. On completing the two sweeps, on each leaf node, the coordinate transformations, the state dependent accelerations, the active joint forces and the state dependent elastic deformation terms are calculated. The active forces are state dependent and include actuator forces acting at the joints, damping forces, body forces like gravity as well as the elastic forces arising from the deformation of the bodies. The final two sweeps correspond to the hierarchic assembly and the hierarchic disassembly processes respectively.

In the analytical treatment presented here, direction cosine matrices and transformation between different basis are not shown explicitly. Appropriate basis transformations must be taken into account for an implementation of this algorithm. Also, this algorithm uses a mixed set of coordinates viz. Cartesian coordinates and relative coordinates throughout the derivation. Mixed set coordinates have been used in [25][26] for rigid body dynamics and in [27] for flexible body dynamics among others.

3 Extension to Flexible Bodies

The following subsections present the details associated with efficiently dealing with flexible bodies within the divide and conquer assemblies, and having these flexible aspects manifest themselves properly through the assembly handles to the rest of the system.

3.1 Kinematic Preliminaries

Consider flexible *Body k* as a typical body of the articulated body system. *Body k* has two handles corresponding to the two joints by which it is connected to the rest of the system. As with the basic development already presented, this procedure is easily extendable to more than two handles. As before, the inward joint is J^k with associated handle k^+, and the outward joint is J^{k+1} with associated handle $k+1^-$. The local "body fixed" reference frame associated with $Body\ k$, is rigidly attached to the material point of $Body\ k$, located at joint handle k^+, and thus rotates and translates with that material point. *Body k* undergoes relative rigid body motion at the joint handle k^+ with respect to its inward (parent) body and deforms elastically with respect to its body fixed reference frame.

Figure (3) shows the deformed and undeformed configurations of the *Body k*. In its undeformed configuration let P_0 be an arbitrary differential volume δD and let the vector $\vec{r_0}$ describe the position of P_0 with respect to the body fixed reference frame at J^{k+} (k^+). After undergoing elastic deformation, let P denote the differential volume in the deformed configuration. $\vec{r}_P$ is the vector describing the position of point P with respect to k^+. The displacement vector $\vec{\delta}$ is expressed in terms of shape functions evaluated at P, $\phi_i^k|_P$, and modal coordinates q_i^k. Finally, ν_k is the number of mode shapes associated with *Body k*.

$$\vec{r}_P = \vec{r_0} + \vec{\delta} \tag{17}$$

$$\vec{\delta} = \sum_{i=1}^{\nu_k} \vec{\varphi}_i^{\,k} q_i^k|_P \tag{18}$$

Let V^{k^+} represent the spatial velocity of handle (point) k^+ with respect to the inertial reference frame N. The kinematic expressions for velocity are obtained as below.

$$V^{k^+} = \begin{bmatrix} {}^N\vec{\omega}^{k^+} \\ {}^N\vec{v}^{k^+} \end{bmatrix}_{(6\times1)} \tag{19}$$

$$V^P = \begin{bmatrix} {}^N\vec{\omega}^{P} \\ {}^N\vec{v}^{P} \end{bmatrix}_{(6\times1)} = {}^NV^{k^+} + \begin{bmatrix} \vec{0} \\ {}^N\vec{\omega}^{k^+} \times (\vec{r_0} + \vec{\delta}) \end{bmatrix} + \Phi^k \dot{q}^k \tag{20}$$

$$= (S^{P/k^+})^T\, V^{k^+} + \Phi^k \dot{q}^k \tag{21}$$

$$\tag{22}$$

$$\text{where} \quad \Phi^k \dot{q}^k = \sum_{i=1}^{v_k} \phi_i^k \dot{q_i}^k|_P \tag{23}$$

$$\text{with} \quad \phi_i^k = \begin{bmatrix} \vec{\psi}_i^k \\ \vec{\varphi}_i^k \end{bmatrix} \tag{24}$$

$$\text{and} \quad S^{P/k^+} = \begin{bmatrix} \underline{U} & \vec{r}_P \times \\ \underline{0} & \underline{U} \end{bmatrix}_{(6\times 6)} \tag{25}$$

Similarly, let A^{k^+} represent the spatial acceleration of handle k^+ with respect to the inertial reference frame *N*. The kinematic expression for the spatial accelerations of arbitrary differential volume at P can be obtained as below with the understanding that summation is implied over the index *i* unless explicitly stated.

$$A^{k^+} = \begin{bmatrix} {}^N\vec{\alpha}^{k^+} \\ {}^N\vec{a}^{k^+} \end{bmatrix}_{(6\times 1)} \tag{26}$$

$$A^P = (S^{P/k^+})^T A^{k^+} + A_t^{P/k^+} + \Phi^k \ddot{q}^k \tag{27}$$

$$\text{where} \quad \Phi^k \ddot{q}^k = \sum_{i=1}^{\nu_k} \phi_i{}^k \ddot{q_i}^k|_P \tag{28}$$

$$\text{and} \quad A_t^{P/k^+} = \begin{bmatrix} {}^N\vec{\omega}^{k^+} \times \sum_{i=1}^{\nu_k} \psi_i{}^k \dot{q_i}^k \\ {}^N\vec{\omega}^{k^+} \times ({}^N\vec{\omega}^{k^+} \times \vec{r}_P) + 2{}^N\vec{\omega}^{k^+} \times \sum_{i=1}^{\nu_k} \vec{\varphi}_i{}^k \dot{q_i}^k \end{bmatrix}_P \tag{29}$$

This formulation uses a *mixed* set of state variables (i.e. a redundant set of both absolute and relative variables) to describe both the spatial velocities and spatial accelerations. As such the spatial velocity for handle k^+ may be correctly described using absolute generalized speed u^{k^+} as

$$V^{k^+} = \begin{bmatrix} {}^N\vec{\omega}^{k^+} \\ {}^N\vec{v}^{k^+} \end{bmatrix} \tag{30}$$

$$= \begin{bmatrix} {}^N\vec{\omega}_r^{k^+} \\ {}^N\vec{v}_r^{k^+} \end{bmatrix} u_r^{k+} + \begin{bmatrix} {}^N\vec{\omega}_t^{k^+} \\ {}^N\vec{v}_t^{k^+} \end{bmatrix} \tag{31}$$

$$= P_r^{k+} u_r^{k+} + \underbrace{V_t^{k^+}}_{0} = P^{k+} u^{k+} \tag{32}$$

where summation is implied of the repeated index r $(r = 1, \ldots, 6)$. In this expression, P_r^{k+} represents the r^{th} spatial partial velocity of handle k^+ in reference frame N, with $u_r^{k^+}$ being the associated absolute generalized speed. These generalized speeds have been chosen to be the full set of six measure numbers [28] associated with the absolute velocity and angular velocity of handle k^+ in N. The matrices $P_{(6\times 6)}^{k+}$ and $u_{(6\times 1)}^{k+}$, appearing in the final term of equation(32) thus contain the complete set spatial partial velocities, and associated generalized speeds for the describing the absolute spatial velocity of handle k^+. The matrix $V_t^{k^+}$ nominally contains the spatial velocity remainder terms which are associated with any prescribed motion for handle k^+. However, because P^{k+} and u^{k+} used here span the full six degrees-of-freedom of handle k^+ in N, $V_t^{k^+}$ is identically zero.

The spatial velocity of handle k^+ in N may be equally well represented in terms of the spatial velocity of its parent handle k^-, and the relative spatial velocity of k^+ with respect to k^-

as permitted by joint J^k. Specifically,

$$V^{k^+} = V^{k^-} + V^{k^+/k^-} \tag{33}$$
$$= V^{k^-} + (P^{J^k} u^{J^k} + V_t^{J^k}) \tag{34}$$

Here P^{J^k} and u^{J^k} are the matrices containing the spatial partial velocities and associated generalized speeds which fully define the manner and magnitude, respectively, with which free motions may take place across joint J^k. As such P^{J^k} forms a joint *free-modes of motion map* for joint J^k, and $V_t^{J^k}$ contains all terms associated with specified motions handle k^+ relative k^-, occurring across joint J^k.

From equations (19), (31) and (32) the partial velocities for absolute spatial motion of handle k^+ in N can be expressed as

$$P_r^{k+} = \begin{bmatrix} {}^N\vec{\omega}_r^{k^+} \\ {}^N\vec{v}_r^{k^+} \end{bmatrix}_{(6\times1)} \tag{35}$$
$$= \begin{bmatrix} \hat{k}_i \\ \vec{0} \end{bmatrix}_{(6\times1)} \text{for } r = i,\ i = 1,2,3 \tag{36}$$
$$= \begin{bmatrix} \vec{0} \\ \hat{k}_i \end{bmatrix}_{(6\times1)} \text{for } r = i+3,\ i = 1,2,3 \tag{37}$$

Similarly, the partial velocities ${}^N\vec{v}_r^P$ and ${}^N\vec{\omega}_r^P$ associated with differential mass P can be expressed as

$$P_r^P = (S^{P/k^+})^T (P_r^{k+}) + \Phi_r^k|_P \tag{38}$$
$$= \begin{bmatrix} \hat{k}_i \\ \hat{k}_i \times \vec{r}_P \end{bmatrix}_{(6\times1)} \text{for } r = i,\ i = 1,2,3 \tag{39}$$
$$= \begin{bmatrix} \vec{0} \\ \hat{k}_i \end{bmatrix}_{(6\times1)} \text{for } r = i+3,\ i = 1,2,3 \tag{40}$$
$$= \begin{bmatrix} \vec{\psi}_i^k \\ \vec{\varphi}_i^k \end{bmatrix}_{P(6\times1)} \text{for } r = i+6,\ i = 1,\cdots,\nu_k \tag{41}$$

Another entity which is useful in the derivation of this algorithm is the joint J^k orthogonal complement map. The free-modes of motion permitted by the joint and the joint degrees of freedom lie in the space spanned by the column vector(s) of matrix P^{J^k}. It can be interpreted as the $6 \times m^k$ matrix that maps the m^k generalized speeds associated with the relative motions permitted by the joint into a 6×1 spatial relative velocity occurring across the joint. Let D^{J^k} be the orthogonal complement of the joint free-motion subspace matrix P^{J^k}. While P^{J^k} is a $6 \times m^k$ matrix corresponding to the $m^k \times 1$ column of joint degrees of freedom, D^{J^k} is a $6 \times (6 - m^k)$ matrix that maps the directions imposed on the constrained degrees of freedom (which include specified motions) of the joint. As such D^{J^k} exactly spans the space of all constraint loads which may be imposed by joint J^k. For example, with a spherical joint, the translational degrees of motion are constrained while the rotational degrees of freedom are maintained. Hence the corresponding maps are given by

$$P^{J^k} = \begin{bmatrix} \hat{k}_1 & \hat{k}_2 & \hat{k}_3 \\ \vec{0} & \vec{0} & \vec{0} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad D^{J^k} = \begin{bmatrix} \vec{0} & \vec{0} & \vec{0} \\ \hat{k}_1 & \hat{k}_2 & \hat{k}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{42}$$

By definition of the orthogonal complement P^{J^k} and D^{J^k} satisfy the following relation:

$$(D^{J^k})^T P^{J^k} = 0 \tag{43}$$

3.2 Flexible Body Kinetics

The equations of motion of a generic body of the system, *Body k*, can be written as below using a velocity projection formulation also known as Kane's [28] method.

$$\int_{B_k} \{ {}^N\vec{v}_r^{\,P} \cdot ({}^N\vec{a}^P \rho \, d\mathcal{D}) + {}^N\vec{\omega}_r^{\,P} \cdot \overbrace{({}^N\vec{\alpha}^P \, dI)}^{0} - [{}^N\vec{v}_r^{\,P} \cdot \vec{f}^P + {}^N\vec{v}_r^{\,k^+} \cdot \vec{f}_c^{\,k^+} + {}^N\vec{\omega}_r^{\,k^+} \cdot \vec{\tau}_c^{\,k^+}$$
$$+ {}^N\vec{v}_r^{\,k+1^-} \cdot \vec{f}_c^{\,k+1^-} + {}^N\vec{\omega}_r^{\,k+1^-} \cdot \vec{\tau}_c^{\,k+1^-}]\} \quad = 0 \tag{44}$$

Consider the constituent terms of the above equation separately. In the above equation $\int_{B_k} {}^N\vec{v}_r^{\,P} \cdot ({}^N\vec{a}^P \rho \, d\mathcal{D})$ is the Generalized Inertia Force term, where ${}^N\vec{a}^P$ can be expanded as below.

$$\begin{aligned} \int_{B_k} {}^N\vec{v}_r^{\,P} \cdot ({}^N\vec{a}^P \rho \, d\mathcal{D}) \;&=\; \int_{B_k} {}^N\vec{v}_r^{\,P} \cdot [\, {}^N\vec{a}^{k^+} + {}^N\vec{\alpha}^{k^+} \times \vec{r}_P + {}^N\vec{\omega}^{k^+} \times ({}^N\vec{\omega}^{k^+} \times \vec{r}_P) \\ &\quad + 2\, {}^N\vec{\omega}^{k^+} \times \vec{\varphi}_i^{\,k} \dot{q}_i^{\,k}|_P + \varphi_i^{\,k} \ddot{q}_i^{\,k}|_P \,]\rho \, d\mathcal{D} \end{aligned} \tag{45}$$

where summation is implied over the repeated indices. For $r = i = 1, 2, 3$, associated with the spatial rotation of the *Body k* parent joint, the above becomes

$$\begin{aligned} \int_{B_k} {}^N\vec{v}_r^{\,P} \cdot ({}^N\vec{a}^P \rho \, d\mathcal{D}) &= [\int_{B_k} \vec{r}_P \times (\hat{k}_i \times \vec{r}_P)\rho d\mathcal{D})] \cdot {}^N\vec{\alpha}^{k^+} + \hat{k}_i \cdot [\int_{B_k} \vec{r}_P \rho d\mathcal{D} \times {}^N\vec{a}^{k^+}] \\ -\hat{k}_i \cdot [{}^N\vec{\omega}^{k^+} \times \int_{B_k} (\vec{r}_P \vec{r}_P \rho d\mathcal{D}) \cdot {}^N\vec{\omega}^{k^+}] &+ 2\, {}^N\vec{\omega}^{k^+} \cdot [\int_{B_k} \vec{\varphi}_j^{\,k}|_P \times (\hat{k}_i \times \vec{r}_P)\rho d\mathcal{D}] \dot{q}_j^{\,k} \\ &+ [\hat{k}_i \cdot \int_{B_k} (\vec{\varphi}_j^{\,k}|_P \times \vec{r}_P \rho d\mathcal{D})] \ddot{q}_j^{\,k} \end{aligned} \tag{46}$$

For $r = i + 3, (i = 1, 2, 3)$, associated with the spatial translation of the *Body k* parent joint, the expansion becomes

$$\int_{B_k} {}^N\vec{v}_r^{\,P} \cdot ({}^N\vec{a}^P \rho \, d\mathcal{D}) = \hat{k}_i \cdot ({}^N\vec{\alpha}^{k^+} \times \int_{B_k} \vec{r}_P \rho d\mathcal{D}) + \hat{k}_i \cdot [{}^N\vec{\omega}^{k^+} \times ({}^N\vec{\omega}^{k^+} \times \int_{B_k} \vec{r}_P \rho d\mathcal{D})]$$
$$+ \hat{k}_i \cdot \int_{B_k} (\rho \vec{\varphi}_j^{\,k}|_P d\mathcal{D}) \ddot{q}_j^{\,k} + \hat{k}_i \cdot ({}^N\vec{a}^{k^+} \int_{B_k} \rho d\mathcal{D}) + \hat{k}_i \cdot [2\, {}^N\vec{\omega}^{k^+} \times \int_{B_k} (\rho \vec{\varphi}_j^{\,k}|_P d\mathcal{D})] \dot{q}_j^{\,k} \tag{47}$$

For $r = i + 6, (i = 1, \cdots, \nu_k)$, associated with the modal coordinates of *Body k*, it takes the form

$$\int_{B_k} {}^N\vec{v}_r^{\,P} \cdot ({}^N\vec{a}^P \rho \, d\mathcal{D}) = {}^N\vec{a}^{k^+} \int_{B_k} (\vec{\varphi}_i^{\,k}|_P \rho d\mathcal{D}) + {}^N\vec{\alpha}^{k^+} \int_{B_k} (\vec{r}_P \times \vec{\varphi}_i^{\,k}|_P)\rho d\mathcal{D}$$
$$+ \int_{B_k} (\vec{\varphi}_i^{\,k}|_P \cdot \vec{\varphi}_j^{\,k}|_P \rho d\mathcal{D})\ddot{q}_j^{\,k} + {}^N\vec{\omega}^{k^+} \cdot \int_{B_k} ([\vec{r}_P \vec{\varphi}_i^{\,k}|_P - \vec{\varphi}_i^{\,k}|_P \vec{r}_P]\rho d\mathcal{D}) \cdot {}^N\vec{\omega}^{k^+}$$
$$+ \int_{B_k} [\vec{\varphi}_i^{\,k}|_P \cdot (2\, {}^N\vec{\omega}^{k^+} \times (\rho \vec{\varphi}_j^{\,k}|_P))]\dot{q}_j^{\,k} d\mathcal{D} \tag{48}$$

The above equations can be collected together and expressed in matrix format as

$$\int_B {}^N\vec{v}_r^{\,P} \cdot ({}^N\vec{a}^P \rho \, d\mathcal{D}) = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} \\ \Gamma_{21} & \Gamma_{22} & \Gamma_{23} \\ \Gamma_{31} & \Gamma_{32} & \Gamma_{33} \end{bmatrix}^k \begin{bmatrix} \alpha^{k+} \\ a^{k+} \\ \ddot{q}^k \end{bmatrix} + \begin{bmatrix} \beta_{11} \\ \beta_{12} \\ \beta_{13} \end{bmatrix}^k \tag{49}$$

A closer inspection of above equations (46-48) clearly identifies several terms in the mass matrix which retain the coupling between the finite rotations at the joints and the elastic deformation of the body. This clearly shows the mass matrix to be a nonlinear function of the joint and flexible body coordinates while the bias force is a nonlinear function of the coordinates and the speeds.

The equations above contain integrals over the volume of the body. Once the modal set Φ^k to be used for each of the flexible bodies has been selected, these spatial integrals (or summations in the case of discrete masses) produce time invariant coefficients for the temporally varying quantities $(q, \dot{q}, \ddot{q})$. As such these coefficients need only be calculated once at the beginning (a pre-processing step) of the simulation.

From equation (44), the *Generalized Constraint Force* contribution arising from the handle k^+ is given by the $[{}^N\vec{v}_r^{\,k^+} \cdot \vec{f}_c^{\,k^+} + {}^N\vec{\omega}_r^{\,k^+} \cdot \vec{\tau}_c^{\,k^+}]$ term. For $r = i = 1, 2, 3$ it is expanded as

$$({}^N\vec{v}_r^{\,k^+} \cdot \vec{f}_c^{\,k+} + {}^N\vec{\omega}_r^{\,k^+} \cdot \vec{\tau}_c^{\,k+}) \;=\; \hat{k}_i \cdot [(\vec{f}_c^{\,k+} \times \vec{r}_P) + \vec{\tau}_c^{\,k+}] = {\gamma_1}^{k^+} {F_c}^{k+1} \tag{50}$$

while for $r = i + 3, (i = 1, 2, 3)$ it is expressed as

$$({}^N\vec{v}_r^{\,k^+} \cdot \vec{f}_c^{\,k+} + {}^N\vec{\omega}_r^{\,k^+} \cdot \vec{\tau}_c^{\,k+}) \;=\; \hat{k}_i \cdot \vec{f}_c^{\,k+} = {\gamma_2}^{k^+} {F_c}^{k+} \tag{51}$$

And for $r = i + 6, i = 1, \cdots, \nu_k$ it is expressed as

$$({}^N\vec{v}_r^{\,k^+} \cdot \vec{f}_c^{\,k+} + {}^N\vec{\omega}_r^{\,k^+} \cdot \vec{\tau}_c^{\,k+}) = [\vec{\varphi}_i^{\,k}|_{k+} \cdot \vec{f}_c^{\,k+} + \vec{\psi}_i^{\,k}|_{k+} \cdot \vec{\tau}_c^{\,k+}] = {\gamma_3}^{k^+} {F_c}^{k+} \tag{52}$$

In equation (44) $[{}^N\vec{v}_r^{\,k+1^-} \cdot \vec{f}_c^{\,k+1^-} + {}^N\vec{\omega}_r^{\,k+1^-} \cdot \vec{\tau}_c^{\,k+1^-}]$ is the *Generalized Constraint Force* term associated with the handle J^{k+1}. It can be expanded for $r = i = 1, 2, 3$ as

$$({}^N\vec{v}_r^{\,k+1^-} \cdot \vec{f}_c^{\,k+1^-} + {}^N\vec{\omega}_r^{\,k+1^-} \cdot \vec{\tau}_c^{\,k+1^-}) \;=\; \vec{f}_c^{\,k+1^-} \cdot (\hat{k}_i \times \vec{r}^{\,k+1^-}) + \hat{k}_i \cdot \vec{\tau}_c^{\,k+1^-} \tag{53}$$
$$=\; {\gamma_1}^{k+1^-} {F_c}^{k+1^-} \tag{54}$$

while for $r = i + 3, (i = 1, 2, 3)$ it is expressed as

$$({}^N\vec{v}_r^{\,k+1^-} \cdot \vec{f}_c^{\,k+1^-} + {}^N\vec{\omega}_r^{\,k+1^-} \cdot \vec{\tau}_c^{\,k+1^-}) = \hat{k}_i \cdot \vec{f}_c^{\,k+1^-} = {\gamma_2}^{k+1^-} {F_c}^{k+1^-} \tag{55}$$

And for $r = i + 6, i = 1, \cdots, \nu_k$ it is expressed as

$$\begin{aligned}
({}^N\vec{v}_r^{k+1^-} \cdot \vec{f}_c^{k+1^-} + {}^N\vec{\omega}_r^{k+1^-} \cdot \vec{\tau}_c^{k+1^-}) &= [\vec{\varphi}_i^k|_{k+1^-} \cdot \vec{f}_c^{k+1^-} + \vec{\psi}_i^k|_{k+1^-} \cdot \vec{\tau}_c^{k+1^-}] \quad (56)\\
&= {\gamma_3}^{k+1^-} {F_c}^{k+1^-} \quad (57)
\end{aligned}$$

Similarly, in equation (44) $[{}^N\vec{v}_r{}^P \cdot \vec{f}^P]$ is the *Generalized Body Force* term. This term includes the stiffness terms originating from the deformation of the flexible body as well as gravitational and other body forces. It can be expanded as

for $r = i = 1, 2, 3$

$$\int_{B_k} {}^N\vec{v}_r^P \cdot \vec{f}^P d\mathcal{D}) = \hat{k}_i \cdot \int_{B_k} (\vec{f}^P \times \vec{r}_P) d\mathcal{D} = \beta_{21} \quad (58)$$

for $r = i + 3, (i = 1, 2, 3)$

$$\int_{B_k} {}^N\vec{v}_r^P \cdot \vec{f}^P d\mathcal{D} = \hat{k}_i \cdot \int_{B_k} \vec{f}^P d\mathcal{D} = \beta_{22} \quad (59)$$

for $r = i + 6, (i = 1, \cdots, \nu_k)$

$$\int_{B_k} {}^N\vec{v}_r^P \cdot \vec{f}^P d\mathcal{D}) = \int_{B_k} \vec{\varphi}_i^k|_P \cdot \vec{f}^P \, d\mathcal{D} = \beta_{23} \quad (60)$$

Collecting all the above equations into a matrix form, the equations of motion for *Body k* can be expressed as

$$\begin{bmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} \\ \Gamma_{21} & \Gamma_{22} & \Gamma_{23} \\ \Gamma_{31} & \Gamma_{32} & \Gamma_{33} \end{bmatrix}^k \begin{bmatrix} \alpha^{k+} \\ a^{k+} \\ \ddot{q}^k \end{bmatrix} - \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{bmatrix}^{k+} F_c^{k^+} - \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{bmatrix}^{k+1^-} F_c^{k+1^-} + \begin{bmatrix} \beta_{11} - \beta_{21} \\ \beta_{12} - \beta_{22} \\ \beta_{13} - \beta_{23} \end{bmatrix}^k = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (61)$$

The above matrix equations can be further consolidated in terms of the rigid body joint coordinates (variables) and the modal coordinates as

$$\begin{bmatrix} \Gamma_{RR} & \Gamma_{RF} \\ \Gamma_{FR} & \Gamma_{FF} \end{bmatrix}^k \begin{bmatrix} A \\ \ddot{q} \end{bmatrix}^{k+} - \begin{bmatrix} \gamma_R \\ \gamma_F \end{bmatrix}^{k^+} F_c^{k+} - \begin{bmatrix} \gamma_R \\ \gamma_F \end{bmatrix}^{k+1^-} F_c^{k+1^-} + \begin{bmatrix} \beta_R \\ \beta_F \end{bmatrix}^k = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (62)$$

Here the terms with subscripts *RR* and *R* are associated with the absolute rigid body motion of handle k^+. The terms with subscript *FR* corresponds to the coupling between the absolute rigid body motion of handle k^+ and the flexible modes of body k. The terms with subscripts *FF* and *F* are associated solely with the flexible body degrees of freedom of k. The above are two sets of equations in terms of two sets of unknowns $\ddot{q}^k$ and A^{k+}. Solving the lower equation, an expression for $\ddot{q}^k$ can be obtained as shown below.

$$\Gamma_{FR}^k A^{k+} + \Gamma_{FF}^k \ddot{q}^k - \gamma_F^{k+} F_c^{k+} - \gamma_F^{k+1^-} F_c^{k+1^-} + \beta_F^k = 0 \quad (63)$$

$$\Rightarrow \ddot{q}^k = (-\Gamma_{FF}^k)^{-1} \big[\Gamma_{FR}^k A^{k+} - \gamma_F^{k+} F_c^{k+} - \gamma_F^{k+1^-} F_c^{k+1^-} + \beta_F^k\big] \quad (64)$$

Substituting the expression for $\ddot{q}^k$ in equation (63), an expression for A^k can be obtained as

$$\begin{aligned}
&\big[\Gamma_{RR}^k - \Gamma_{RF}^k (\Gamma_{FF}^k)^{-1} \Gamma_{FR}^k\big] A^{k+} - \big[\gamma_R^{k+} - \Gamma_{RF}^k (\Gamma_{FF}^k)^{-1} \gamma_F^{k+}\big] F_c^{k+} \quad (65)\\
&-\big[\gamma_R^{k+1^-} - \Gamma_{RF}^k (\Gamma_{FF}^k)^{-1} \gamma_F^{k+1^-}\big] F_c^{k+1^-} + \big[\beta_R^k - \Gamma_{RF}^k (\Gamma_{FF}^k)^{-1} \beta_F^k\big] = 0
\end{aligned}$$

$$\Rightarrow A^{k+} = \zeta_{11}^k F_c^{k+} + \zeta_{12}^k F_c^{k+1^-} + \zeta_{13}^k \quad (66)$$

The above expressions involve an inversion of the $\mathit{\Gamma}_{FF}^{k}$ matrix. This matrix is local to $Body\ k$, is of dimension $\nu_k \times \nu_k$, and remains constant. It is diagonal if only orthogonal vibration modes are used. Adding static correction modes will introduce off-diagonal terms that produce coupling between the vibration and correction modes. The sparsity of this matrix is thus dependent on the number of vibration and correction modes used. However, in all cases, the $\mathit{\Gamma}_{FF}^{k}$ is a symmetric positive definite matrix and it remains constant through the simulation. The computational cost of inverting this matrix is thus a fixed cost in the pre-processing analysis and there is no repeated cost incurred during the simulation.

Now consider the spatial acceleration of handle $k+1^-$ which is given by

$$A^{k+1^-} = (S^{k/k+1})^T A^{k+} + A_t^{k+1^-/k^+} + \mathit{\Phi}^{k+1/k}\ddot{q}^k \tag{67}$$

Substituting the expression for $\ddot{q}^k$ in the above equation, the equation becomes

$$\begin{aligned} A^{k+1^-} = {} & (S^{k/k+1})^T A^{k+} + A_t^{k+1^-/k^+} - (\mathit{\Phi}^k|_{k+1^-})^T(\mathit{\Gamma}_{FF}^k)^{-1}\big[\mathit{\Gamma}_{FR}^k A^{k+} - \gamma_F^{k+} F_c^{k+} \\ & -\gamma_F^{k+1^-} F_c^{k+1^-} + \beta_F^k\big] \end{aligned} \tag{68}$$

$$\begin{aligned} \Rightarrow A^{k+1^-} = {} & \big[(S^{k/k+1})^T - (\mathit{\Phi}^k|_{k+1^-})^T(\mathit{\Gamma}_{FF}^k)^{-1}\mathit{\Gamma}_{FR}^k\big] A^{k+} + \big[(\mathit{\Phi}^k|_{k+1^-})^T(\mathit{\Gamma}_{FF}^k)^{-1}\gamma_F^{k+1^-}\big] F_c^{k+1^-} \\ & + \big[(\mathit{\Phi}^k|_{k+1^-})^T(\mathit{\Gamma}_{FF}^k)^{-1}\gamma_F^k\big] F_c^{k+} + \big[A_t^{k+1^-/k^+} - (\mathit{\Phi}^k|_{k+1^-})^T(\mathit{\Gamma}_{FF}^k)^{-1}\beta_F^k\big] \end{aligned} \tag{69}$$

$$\Rightarrow A^{k+1^-} = {\eta_1}^k A^{k+} + {\eta_2}^k F_c^{k+} + {\eta_3}^k F_c^{k+1^-} + {\eta_4}^k \tag{70}$$

where η_i $(i = 1:4)$ simply represent useful intermediate coefficients which will aid in subsequent manipulations.

Now substituting the expression for A^k into equation (70), the expression for A^{k+1^-} can be obtained as

$$A^{k+1^-} = [{\eta_2}^k + {\eta_1}^k \zeta_{11}^k] F_c^{k+} + [\eta_3^k + \eta_1^k \zeta_{12}^k] F_c^{k+1^-} + [\eta_4^k + \eta_1^k \zeta_{13}^k] \tag{71}$$
$$= \zeta_{21}^k F_c^{k+} + \zeta_{22}^k F_c^{k+1^-} + \zeta_{23}^k \tag{72}$$

Thus, from equations (66) and (72) the two handle articulated body equations for the flexible *body k* are given by

$$A^{k+} = \zeta_{11}^k F_c^{k+} + \zeta_{12}^k F_c^{k+1^-} + \zeta_{13}^k \tag{73}$$
$$A^{k+1^-} = \zeta_{21}^k F_c^{k+} + \zeta_{22}^k F_c^{k+1^-} + \zeta_{23}^k \tag{74}$$

Similarly for *body (k+1)*, the two handle equations of motion are

$$A^{k+1^+} = \zeta_{11}^{k+1} F_c^{k+1} + \zeta_{12}^{k+1} F_c^{k+2^-} + \zeta_{13}^{k+1} \tag{75}$$
$$A^{k+2^-} = \zeta_{21}^{k+1} F_c^{k+1} + \zeta_{22}^{k+1} F_c^{k+2^-} + \zeta_{23}^{k+1} \tag{76}$$

These equations are now in the same form as that of the two handle articulated body equations of motion found in Featherstone's Divide-and-Conquer algorithm for rigid bodies (RDCA) and as discussed in above section (2.1). These can now be coupled together to form the two handle equations of motion of the resulting assembly.

3.3 Recursive Expression for Inertia Coupling Terms

In the discussion of the general scheme of the Divide and Conquer Algorithm, it was mentioned that a set of recursive formulae is used to couple together the equations of motion of successive bodies in the system to form the equations of motion of the resulting assemblies. Reference [18] derives a set of recursive formulae but recommends an alternate set of formulae (without derivation) for use in actual implementation for better computational efficiency. In this section, a derivation of this alternate set of formulae is presented.

The relative acceleration at the joint connecting *Body k* and *Body k+1* is given by the following equation.

$$A^{k+1^+} - A^{k+1^-} = P^{J^{k+1}}\dot{u}^{J^{k+1}} + A_t^{J^{k+1}} \tag{77}$$

By Newton's third law of motion the constraint force at handle $k+1^+$ viz. $F_c^{k+1^+}$ and handle $k+1^-$ viz. $F_c^{k+1^-}$ are equal in magnitude and opposite in direction. Using this fact and substituting the expressions for ${}^N A^{k+1^-}$ and ${}^N A^{k+1^+}$ from equations (74) and (75) into equation (77), an expression for $F_c^{k+1^+}$ is obtained as

$$\begin{aligned}[\zeta_{11}^{k+1} + \zeta_{22}^{k}]F_c^{k+1^+} = [\zeta_{21}^{k}F_c^{k+} - \zeta_{12}^{k+1}F_c^{k+2^-} + \zeta_{23}^{k} \\ -\zeta_{13}^{k+1} + P^{J^{k+1}}\dot{u}^{J^{k+1}} + A_t^{J^{k+1}}]\end{aligned} \tag{78}$$

$$\begin{aligned}\Rightarrow F_c^{k+1^+} = [\zeta_{11}^{k+1} + \zeta_{22}^{k}]^{-1}[\zeta_{21}^{k}F_c^{k+} - \zeta_{12}^{k+1}F_c^{k+2^-} \\ +\zeta_{23}^{k} - \zeta_{13}^{k+1} + P^{J^{k+1}}\dot{u}^{J^{k+1}} + A_t^{J^{k+1}}]\end{aligned} \tag{79}$$

Pre-multiplying equation (78) by $(D^{J^{k+1}})^T$ gives

$$\begin{aligned}(D^{J^{k+1}})^T[\zeta_{11}^{k+1} + \zeta_{22}^{k}]F_c^{k+1^+} = (D^{J^{k+1}})^T[\zeta_{21}^{k}F_c^{k+} + \zeta_{23}^{k} - \zeta_{13}^{k+1} - \zeta_{12}^{k+1}F_c^{k+2^-} \\ +A_t^{J^{k+1}}] + \underbrace{(D^{J^{k+1}})^T P^{J^{k+1}}}_{0}\dot{u}\end{aligned} \tag{80}$$

From the definition of the orthogonal complement of joint motion subspace, the constraint force $F_c^{k+1^+}$ can be expressed in terms of the measure numbers of the constraint torques $\tilde{\tau}_c{}^{k+1^+}$ and constraint forces $\tilde{f}_c{}^{k+1^+}$ as

$$F_{MN}^{k+1^+} = \begin{bmatrix}\tilde{\tau}_c{}^{k+1^+} \\ \tilde{f}_c{}^{k+1^+}\end{bmatrix} \tag{81}$$

$$F_c^{k+1^+} = D^{J^{k+1}}F_{MN}^{k+1^+} \tag{82}$$

Substituting the above into equation (80)

$$\begin{aligned}(D^{J^{k+1}})^T[\zeta_{11}^{k+1} + \zeta_{22}^{k}]D^{J^{k+1}}F_{MN}^{k+1^+} = (D^{J^{k+1}})^T[\zeta_{21}^{k}F_c^{k+} - \zeta_{12}^{k+1}F_c^{k+2^-} \\ +\zeta_{23}^{k} - \zeta_{13}^{k+1} + A_t^{J^{k+1}}]\end{aligned} \tag{83}$$

The term $(D^{J^{k+1}})^T[\zeta_{11}{}^{k+1^+} + \zeta_{22}{}^{k}]D^{J^{k+1}}$ is a symmetric positive definite matrix and hence there is no problem associated with its inversion.

$$\text{Let} \quad (D^{J^{k+1}})^T[\zeta_{11}^{k+1} + \zeta_{22}^{k}]D^{J^{k+1}} = \widehat{X} \tag{84}$$

$$\Rightarrow F_{MN}^{k+1^+} = \widehat{X}^{-1}(D^{J^{k+1}})^T[\zeta_{21}^{k}F_c^{k+} - \zeta_{12}^{k+1}F_c^{k+2^-} + \zeta_{23}^{k} - \zeta_{13}^{k+1} + A_t^{J^{k+1}}] \tag{85}$$

Pre-multiplying the above expression by $D^{J^{k+1}}$ to get the desired expression for $F_c^{k+1^+}$

$$F_c^{k+1^+} = D^{J^{k+1}} F_{MN}^{k+1^+} \tag{86}$$

$$= D^{J^{k+1}k} \widehat{X}^{-1} (D^{J^{k+1}})^T [\zeta_{21}^k F_c^{k+} - \zeta_{12}^{k+1} F_c^{k+2^-} + \zeta_{23}^k - \zeta_{13}^{k+1} + A_t^{J^{k+1}}] \tag{87}$$

The above expression for $F_c^{k+1^+}$ can be compactly written as below

$$F_c^{k+1^+} = \widehat{W} \zeta_{21}^k F_c^{k+} - \widehat{W} \zeta_{12}^{k+1} F_c^{k+2^-} + \widehat{Y} \tag{88}$$

$$\text{where} \quad \widehat{W} = D^{J^{k+1}} \widehat{X}^{-1} (D^{J^{k+1}})^T \tag{89}$$

$$\text{and} \quad \widehat{Y} = \widehat{W} [\zeta_{23}^k - \zeta_{13}^{k+1} + A_t^{J^{k+1}}] \tag{90}$$

The expression for $F_c^{k+1^-}$ is substituted in equation (73) and (76) and after some algebraic manipulation, the two handle equation of motion of the assembly of *Body k* and *Body (k+1)* are obtained as

$$A^{k+} = [\zeta_{11}^k - \zeta_{12}^k \widehat{W} \zeta_{21}^k] F_c^{k+} + \zeta_{12}^k \widehat{W} \zeta_{12}^{k+1} F_c^{k+2^-} + \zeta_{13}^k - \zeta_{12}^k \widehat{Y} \tag{91}$$

$$A^{k+2^-} = \zeta_{21}^{k+1} \widehat{W} \zeta_{21}^k F_c^{k+} + [\zeta_{22}^{k+1} - \zeta_{21}^{k+1} \widehat{W} \zeta_{12}^{k+1}] F_c^{k+2^-} + \zeta_{23}^{k+1} + \zeta_{21}^{k+1} \widehat{Y} \tag{92}$$

From above, the recursive expression for $\Upsilon_{ij}^{k:k+1}$ appearing in equations (7)(8) can be obtained as

$$\Upsilon_{11}^{k:k+1} = [\zeta_{11}^k - \zeta_{12}^k \, \widehat{W} \, \zeta_{21}^k] \tag{93}$$

$$\Upsilon_{22}^{k:k+1} = [\zeta_{22}^{k+1} - \zeta_{21}^{k+1} \, \widehat{W} \, \zeta_{12}^{k+1}] \tag{94}$$

$$\Upsilon_{12}^{k:k+1} = \zeta_{12}^k \, \widehat{W} \, \zeta_{12}^{k+1} \tag{95}$$

$$\Upsilon_{21}^{k:k+1} = \zeta_{21}^{k+1} \, \widehat{W} \, \zeta_{21}^k \tag{96}$$

$$\Upsilon_{13}^{k:k+1} = \zeta_{13}^k - \zeta_{12}^k \widehat{Y} \tag{97}$$

$$\Upsilon_{23}^{k:k+1} = \zeta_{23}^{k+1} + \zeta_{21}^{k+1} \widehat{Y} \tag{98}$$

These recursive formulae are used in the hierarchic assembly process to couple together the equations of motion of successive assemblies to form the two handle equations of motion of the resulting higher order assembly.

4 Computational Complexity

As indicated in section (2.1), each body of the system represents a single node at the leaf level of a binary tree. The algorithm works in four sweeps of the binary tree and the calculation of the spatial acceleration of the joints is carried out in $O(log(n_b))$ complexity (when performed in parallel) as explained in [18]. The traversal of the system topology in the binary tree form allows the processes to be time optimal $O(log(n_b))$ by using $O(n_b)$ processors. The binary tree is mapped directly onto $O(n_b)$ processors in a tree structure, where n_b in this context is the number of bodies (leaf level subassemblies) which make up the system. The architecture of the process is discussed in [18][19] and is not discussed further here.

Once the spatial accelerations of the handles on a constituent body have been determined, the time derivatives of the modal coordinates for the body are uncoupled and can be calculated independent of the other bodies of the system. As indicated in equation (64), the expression for the time derivatives of the modal coordinates involves a matrix inversion of the term Γ_{FF}^k. If the admissible mode shapes for a body is comprised only of natural modes of vibration which

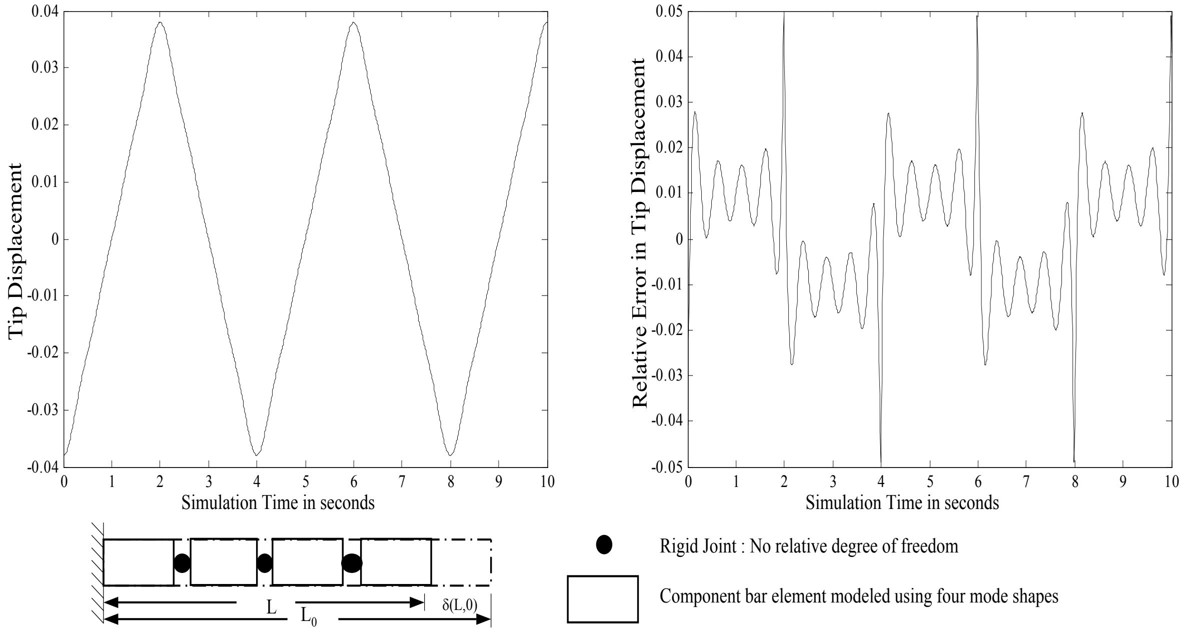


Figure 4: Longitudinal Vibration of Substructured Bar

are mutually orthogonal, this matrix is diagonal. Hence the cost associated with its inversion is $O(\nu_k)$. However if the admissible mode shapes chosen are not orthogonal, as in the case of static correction modes, the matrix is no longer diagonal. Depending on the number of non-orthogonal mode shapes, the matrix is sparsely populated with the off-diagonal terms providing the coupling between non-orthogonal modes. In the worst case where there is coupling between all modes chosen for the body, the matrix is fully populated and the cost associated with its inversion is $O(max(\nu)^3)$. However, the matrix $\mathit{\Gamma}^k_{FF}$ contains time invariant coefficients of the modal degrees of freedom, and hence remains constant during the simulation. Thus its inversion need only be calculated once in a pre-processing step, and as such this cost should not be considered in the cost per temporal integration step determination. The cost which is incurred at each evaluation is the $\Gamma_{RF}\Gamma^{-1}_{FF}$ multiplication which appears in equation (65). This is because matrix Γ_{RF} is state dependent and will thus vary with each temporal integration step. For a fully populated $\mathit{\Gamma}^k_{FF}$, this matrix multiplication yields a maximum cost of $O(\nu_i)^2$ per integration step.

In the binary tree mapping, each body is mapped onto an individual processor. Hence the matrix multiplication $\Gamma_{RF}\Gamma^{-1}_{FF}$ can be independently calculated in $O(\nu_i)^2$ complexity on each processor unless some additional parallel methods are implemented to specifically deal with the possibly large matrix multiplications. These will not be discussed here. Instead, the performance of the method will be crudel determined by assuming that all flexible body manipulations associated with *Body k* are restricted to the single processor to which *Body k* is mapped. The maximum complexity of this process is thus $O(max(\nu_i)^2)$, where $max\ (\nu_i)$ is the maximum number of assumed modes on any constituent body of the system. The effective computational cost (which manifests itself as wall time) of the algorithm can thus be calculated as $O(log(n_b)) + O(max(\nu_i)^2)$ when implemented in parallel on processor optimal $O(n_b)$ processors.

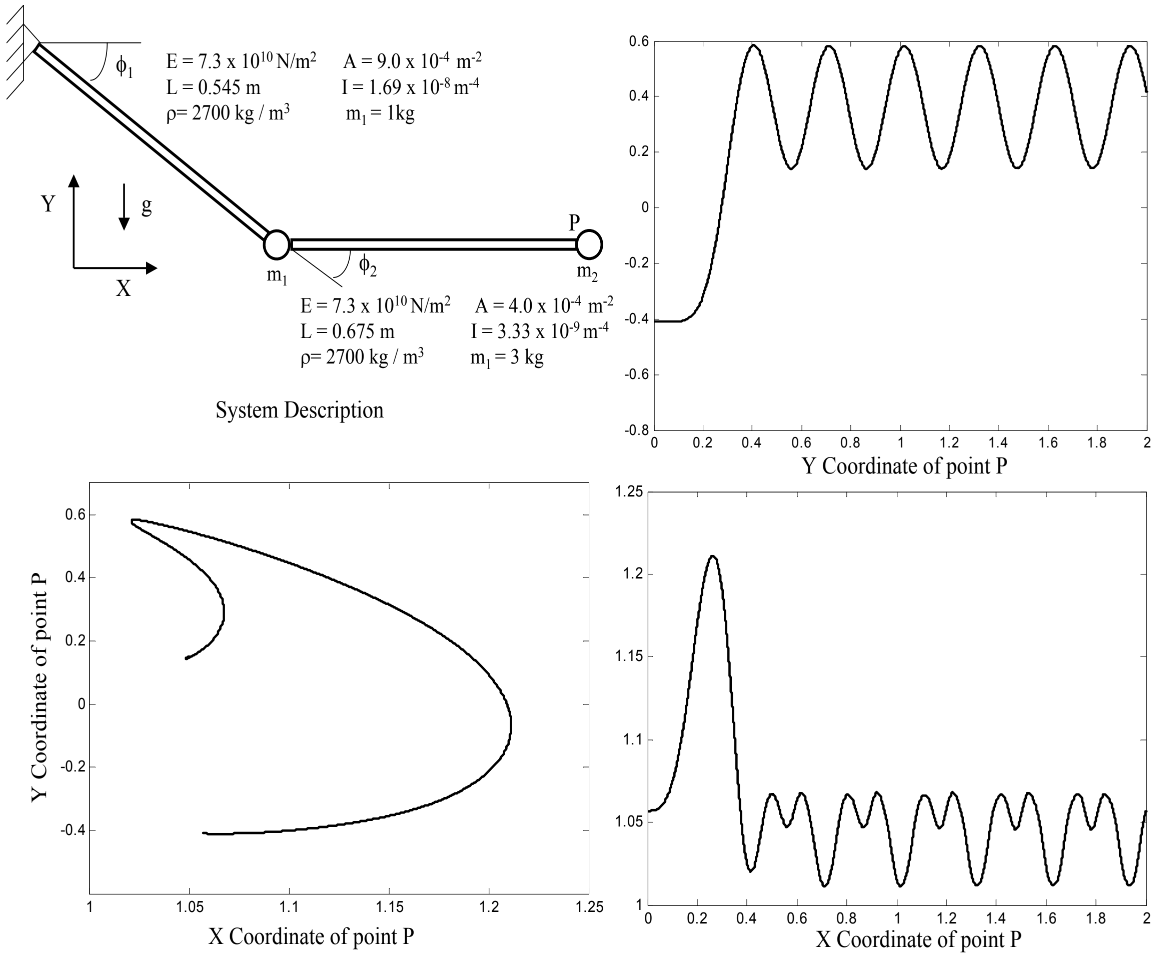


Figure 5: Two Bar Example

5 Numerical Examples

To validate the algorithm presented here, simulation results from the modelling of two test cases are presented here. The first test case is the modelling of the longitudinal vibrations of an uniform bar. The bar is modelled as an elastic body, clamped at one end and free at the other. The modulus of elasticity and mass density of the rod is assumed to be unity. The length of the rod is 4 units and it is released from rest under an initial state of compressive strain of 0.01. The bar is sub-structured into four bar elements, each of unit length. Each bar element is connected to its parent element by rigid joints, i.e. no rigid body degrees of freedom are maintained at the joint between two consecutive bodies. The flexibility of the bar elements is modelled using assumed modes with the first four natural modes of vibration of a clamped-free bar chosen as the admissible shape functions. This system was chosen because the system behavior is easy to visualize and the exact analytical solution is readily available. The result shown in figure (4) is for a 10 second simulation and presents the tip displacement, as well as the displacement error (difference between the analytic solution and the FDCA solution) time histories. The result is in good agreement with the analytical solution, with error of form and magnitude appropriate for the approximation of this continuous system. Since a truncated set of modes are used, the model fails to capture the exact behavior at the tip. This is a characteristic behavior of the assumed mode modelling technique and not a shortcoming of the algorithm. The implementation was carried out in Matlab™ and the Matlab™ $ode45$ was used for numerical integration.

The second example is associated with the articulate flexible two member arm system shown in figure (5). The system consists of two elastic bars, each with point masses at the end. The two joints in the system are revolute and the angular motions of the two bars are prescribed as below described in (99). The transverse and longitudinal vibrations of each arm are modelled using two shape functions, one for each vibration mode as shown in equations (101-102). This system has been simulated in [30]-[32] and established results are available for this problem. The system requires appropriate handling of the geometric stiffening effect and in the implementation here, the modelling approach outlined in [33][34] is used. The system is started from static equilibrium and undergoes prescribed angular motion for 0.5 seconds. The prescribed motion of the angles are shown in equations (99-100). After that, the system performs harmonic oscillations under the effect of gravity and internal strain energy. The results show the variation of the position of the tip (point P) with time. The results presented here are in agreement with the solutions in [30]-[32]. The implementation of the algorithm is clearly able to handle the geometric stiffening effect and accurately capture the dynamics of the system. This implementation too was carried out in Matlab™ and the numerical integration carried out using the Matlab™ $ode45$.

$$\begin{aligned} \phi_1 &= -\pi/4 \cdots -\infty < t < 0 && (99) \\ &= \pi/4(-1 + 72t^3) \cdots 0 < t < 1/6 \\ &= \pi/4(-18t + 108t^2 - 144t^3) \cdots 1/6 \leq t < 1/3 \\ &= \pi/4(-8 + 54t - 108t^2 + 72t^3) \cdots 1/3 \leq t < 1/2 \\ &= \pi/4 \cdots t > 1/2 \\ \phi_2 &= -\phi_1 && (100) \\ \text{Longitudinal Vibration} &: \quad (\frac{x}{L})^2 && (101) \\ \text{Transverse Vibration} &: \quad 1.5(\frac{x}{L})^2 - 0.5(\frac{x}{L})^3 && (102) \end{aligned}$$

6 Conclusion and Future Work

A new algorithm for solving the equations of motion of articulated flexible body systems is presented in this paper. The equations of motion for tree topology articulated body systems comprising of arbitrary number of flexible and rigid bodies connected together by kinematic joints can be efficiently generated and solved using this algorithm. The computational complexity of this algorithm when implemented on $O(n_b)$ processors is expected to be $O(log(n_b))$ $+O(max(\nu_i)^2)$, where max ν_i is the maximum number of admissible shape functions for any constituent body of the system. The algorithm follows a divide and conquer scheme similar to the one presented in [18]. The elastic deformation of the component bodies is modelled by using superposition of a truncated set of admissible shape functions. Other than the use of superposition of a truncated set of component modes, no approximations are made. If the elastic deformations in the algorithm are neglected, the algorithm reduces to an exact algorithm for rigid body articulated systems. Although the equations derived above are for chain systems, the extension to tree topologies is trivial. [19] contains a detailed discussion on the accuracy of the RDCA and suggests a pivoting scheme for improving accuracy in a rigid body context. The discussions in [19] are generic to the DCA scheme and it is expected that they would be applicable to the present work. A detailed analysis and comparison of the numerical accuracy of the algorithm, as well as the implementation of this algorithm for systems with kinematically closed loops are discussed in a forthcoming paper.

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REFERENCES

[1] Hooker W. W. and Margulies G., 1965. “The dynamical attitude equations for an n-body satellite”. Journal of the Astronautical Sciences, 7(4), pp. 123:128, Winter.

[2] Walker M. W. and Orin D. E., 1982. “Efficient dynamic computer simulation of robotic mechanisms”. Journal of Dynamic Systems, Measurement, and Control, 104, pp. 205-211, Sep.

[3] Armstrong W. W., 1979. “Recursive Solution to the Equations of Motion of an N-Link Manipulator”. Fifth World Congress on the Theory of Machines and Mechanisms, **2**, pp. 1342–1346.

[4] Featherstone R. 1983 “The calculation of robotic dynamics using articulated body inertias”. International Journal of Robotics Research, 2(1), pp. 13:30, Spring.

[5] Brandl H., Johanni R. and Otter M., 1986. “A Very Efficient Algorithm for the Simulation of Robots and Similar Multibody Systems Without Inversion of the Mass Matrix”. IFAC/IFIP/IMACS Symposium, Vienna, Austria, pp. 95–100.

[6] Bae D. S., and Haug E. J. 1987. “A recursive formation for constrained mechanical system dynamics: Part I, Open loop systems”. Mechanisms, Structures, and Machines, 15(3), pp. 359:382.

[7] Kurdila A. J., Menon R. G. and Sunkel J. W. 1993. “Nonrecursive Order N Formulation of Multibody Dynamics”. Journal of Guidance, Control, and Dynamics, **16**, (5), pp. 838–844, Sept–Oct.

[8] Anderson K. S., 1995. “An Order-N formulation for the motion simulation of general multi-rigid-body tree system”. Computers and Structures, **46** (3), pp. 547–559.

[9] Jain A., 1991. “Unified Formulation of Dynamics for Serial Rigid Multibody Systems”. Journal of Guidance, Control and Dynamics, **14** (3), pp. 531–542.

[10] Kim S. S. and Haug E. J. 1988. “Recursive Formulation for Flexible Multibody Dynamics: Part I, Open-Loop Systems”. Computer Methods in Applied Mechanics and Engineering, **71**, (3), pp. 293–314.

[11] Jain A. K., Rodriguez G., 1992. “Recursive flexible multibody system dynamics using spatial operators”. Journal of Guidance, Control and Dynamics, **15** (6), pp. 1453–1466.

[12] Anderson K. S., 1995. “Efficient Modelling of General Multibody Dynamic Systems with Flexible Components”. Computational Dynamics in Multibody Systems, (), pp. 79–97. Kluwer Academic Publishers, Printed in Netherlands.

[13] Bauchau O. A., 2003. “Formulation of Modal-Based Elements in Nonlinear, Flexible Multibody Dynamics”. International Journal of Multiscale Computational Engineering, **1** (2-3), pp. 161–180.

[14] Banerjee A. K., 1993. “Block-diagonal equations for multibody elastodynamics with geometric stiffness and constraints”. Journal of Guidance, Control and Dynamics, **16** (6), pp. 1092–1100.

[15] Shabana A. A., 1997. “Flexible Multibody Dynamics: Review of Past and Recent Developments”. Multibody System Dynamics, **1**, pp. 189–222.

[16] Quinn M., 1994. *Parallel Computing, Theory and Practice*. McGraw-Hill, New York.

[17] Fijany A., Shraf I. D’Eleuterio G. M. T. 1995. “Parallel O(log N) computation of manipulator forward dynamics”. IEEE Trans. Robotics and Automation, **11** (3), pp. 389–400.

[18] Featherstone R., 1999. “A Divide-and-Conquer Articulated-Body Algorithm for Parallel O(log(n)) Calculation of Rigid Body Dynamics. Part 1: Basic Algorithm”. The International Journal of Robotic Research, **18** (3), pp. 867–875.

[19] Featherstone R., 1999. “A Divide-and-Conquer Articulated-Body Algorithm for Parallel O(log(n)) Calculation of Rigid Body Dynamics. Part 2: Trees, Loops and Accuracy”. The International Journal of Robotic Research, **18** (3), pp. 876–892.

[20] Shabana A. A., 1996. “Resonance Conditions and Deformable Body Coordinate Systems”. Journal of Sound and Vibration, **192** (1), pp. 389–398.

[21] Schwertassek R., Wallrapp O. and Shabana A. 1999. “Flexible Multibody Simulation and Choice of Shape Functions”. Nonlinear Dynamics, **20**, pp. 361–380.

[22] Schwertassek R., Dombrowski S. and Wallrapp O. 1999. “Modal Representation of Stree in Flexible Multibody Simulations”. Nonlinear Dynamics, **20**, pp. 381–399.

[23] Roberson R. E. and Schwertassek R., 1988 “Dynamics of Multibody Systems”. Springer-Verlag, Berlin

[24] Shabana A. A., 1993. “Substructuring in Flexible Multibody Dynamics”. Proceedings of Computer Aided Analysis of Rigid and Flexible Mechanical Systems: NATO-Advanced Study Institute, **1**, pp. 377–396.

[25] Kim S. S., and VanderPloeg M. J., 1986. “Generalized and Efficient Method for Dynamic Analysis of Mechanical Systems Using Velocity Transforms”. Journal of Mechanism, Transmissions, and Automation Design, **108** (2), pp. 176–182.

[26] Nikravesh P. E., 1990. “Systematic Reduction of Multibody Equations to a Minimal Set”. International Journal of Non-Linear Mechanics, **25** (2-3), pp. 143–151.

[27] Ambrosia J. A. C., 2001. “Complex Flexible Multibody Systems with Application to Vehicle Dynamics”. Multibody System Dynamics **6** (2), pp. 163–182.

[28] Kane T. R. and Levinson D. A., 1985. *Dynamics: Theory and Application*. Mcgraw-Hill, New York.

[29] Mukherjee R., and Anderson K. S., 2006. "Orthogonal Complement Based Divide-and-Conquer Algorithm for Constrained Multibody Systems". Nonlinear Dynamics, In Press.

[30] Botz M. and Hagedorn P., 1997. "Dynamic Simulation of multibody systems including planar elastic beams using Autolev". Engineering Computations, **14** (4), pp. 456–470.

[31] Claus H., 2001. "A Deformation Approach to Stress Distribution in Flexible Multibody Systems". Multibody System Dynamics, **6**, pp. 143–161.

[32] Seo S. and Yoo H., 2002. "Dynamic Analysis of Flexible Beams Undergoing Overall Motion Employing Linear Strain Measures". AIAA Journal, **40** (2), pp. 319–326.

[33] Yoo H., Ryan R. and Scott R., 1995. "Dynamics of Flexible Beams Undergoing Overall Motion". Journal of Sound and Vibration, **181** (2), pp. 261–278.

[34] Yoo H., and Shin S., 1998. "Vibration Analysis of Rotating Cantilever Beams". Journal of Sound and Vibration, **212** (5), pp. 807–828.

[35] Featherstone R., 1987. *Robot Dynamics Algorithms*. Kluwer Academic Publishers, New York.

[36] Shabana A. A., 1998. *Dynamics of Multibody Systems - Second Edition*. Cambridge University Press, New York.

[37] Anderson K. S., 1990. *Recursive Derivation of Explicit Equations of Motion for Efficient Dynamic/Control Simulation of Large Multibody Systems*. PhD Thesis, Stanford University, Stanford, CA, May.

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